Equality of opportunity versus equality of opportunity sets*

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Abstract

We characterize two different approaches to the idea of equality of opportunity. Roemer's social ordering is motivated by a concern to compensate for the effects of certain (non-responsibility) factors on outcomes. Van de gaer's social ordering is concerned with the equalization of the opportunity sets to which people have access. We show how different invariance axioms open the possibility to go beyond the simple additive specification implied by both rules. This offers scope for a broader interpretation of responsibility-sensitive egalitarianism.

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1 Introduction

The traditional idea of equality of opportunity has been interpreted in two different ways in the recent literature. A first approach starts from the premise that individual outcomes are determined by two types of factors: compensation factors (such as innate abilities, handicaps, called "type" in the sequel) and responsibility factors (such as ambition, preferences, called "effort" in the sequel).[1] The basic idea of equality of opportunity then is to equalize individual outcomes to the extent that they reflect differences in "type", while allowing for different outcomes whenever they are due to "effort". This focus on compensation of outcomes is prominent in the larger part of the axiomatic literature —see, e.g., Bossert (1995), Fleurbaey (1994, 1995a, 1995b), Bossert and Fleurbaey (1996), Iturbe-Ormaetxe (1997), Maniquet (1998, 2002). It is also the basic inspiration of Roemer's rule (Roemer, 1993, 1998, 2002).

A second approach to equality of opportunity focuses on the opportunity set to which people have access, and tries to make these sets as equal as possible —see, e.g., Kranich (1996), and

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[1] Roemer (2002) calls these "arbitrary" and "responsible" factors.

Ok and Kranich (1998). Compensation is defined in terms of opportunity sets. The concern for responsibility implies that only the set matters, while individuals remain responsible for their choice. A rule which starts from this inspiration has been proposed by Van de gaer (1993) and is further explored in Bossert, Fleurbaey and Van de gaer (1999).

Of course the two approaches are related, since the opportunity set to which a person with a given type has access gives rise to the vector of possible outcomes that he can obtain for different values of effort. Yet their basic inspiration is different.

Roemer's approach has been very popular in the recent past and it has resulted in a series of thought provoking empirical applications (Llavador and Roemer, 2001, Roemer *et al.*, 2002, Roemer, 2002). As a matter of fact, in his recent "progress report", Roemer (2002) explicitly criticizes the alternative "opportunity set" approach, claiming that it has not resulted in any empirical work because it remains too abstract and fails to take sufficient cues from popular and long-standing conceptions of equality of opportunity. This criticism is certainly not valid for the Van de gaer rule. In many cases, both the Roemer rule and the Van de gaer rule lead to exactly the same policy prescriptions. This happens to be true in all the empirical applications referred to earlier. In cases where the rules differ, the Van de gaer rule is even computationally simpler than the Roemer rule, at least in a continuous setting, because the latter has to integrate the lower countour set of the different types. Empirical applicability therefore cannot be an argument to discard the former.

In this paper, we want to explore in more detail the differences between the Roemer rule and the Van de gaer rule. We briefly review both rules in section 2. We illustrate how Roemer's rule is an example of the CO (*compensating outcomes*) approach, while Van de gaer's rule is an example of the CS (*compensating sets*) approach. Moreover, both rules assume an additive aggregation procedure. In section 3, we introduce notation and present both basic rules, together with some extensions. Section 4 discusses the axioms and section 5 characterizes and discusses the basic versions of the Roemer rule and the Van de gaer rule. The axioms used in the characterization reveal the strong and weak points of both. They also yield a better insight into the implications of the additive aggregation rule underlying both social orderings. In section 6 we show how relaxing this additivity assumption paves the way for a characterization of some alternative rules (also proposed in Kolm, 2001). Section 7 concludes.

2 Two alternative rules

To set the scene, let us first briefly summarize both approaches in a finite setting. Let X be the set of social options, e.g. the different policy prescriptions. Individuals who are homogeneous with respect to compensation factors are of the same "type"; we gather the different types $i = 1, \ldots, m$ in a set M. Individuals who are homogeneous with respect to responsibility characteristics have exerted the same "effort"; the different effort levels $j = 1, \ldots, n$ are collected in a set N. We assume that effort can be observed, either directly (for example, labour hours) or indirectly (for example, via Roemer's percentile approach: on the basis of the rank order of an individual within the distribution of outcomes of his type).[2] To simplify our exposition, we assume that each type-effort couple $(i, j) \in M \times N$ is represented by exactly

[2] Roemer proposes to identify effort on the basis of the quantiles of the outcome distribution for each type. For a critical assessment of this quantile hypothesis, see Fleurbaey (1998) and Kolm (2001). For a strong defence, see Roemer (2002).

one individual; we abbreviate $ij \in MN$. Later on, we mention how to adapt the rules when relaxing this assumption. Outcomes are defined by the social option, the type, and the effort of the individual. Let $U = \left(U_1^1, U_1^2, \ldots, U_m^n\right)$ be a profile of real-valued outcome functions U_i^j, one for each type-effort couple ij, defined as

$$U_i^j : X \to \mathbb{R} \text{ (or } \mathbb{R}_{++}) : x \mapsto U_i^j (x) .$$

To illustrate both approaches, we consider an economy with two types and five effort levels. Figure 1 presents the outcomes of both types under social option x for the different effort levels, i.e. the vectors $U_i (x) = \left(U_i^1 (x), \ldots, U_i^5 (x)\right), i = 1, 2$.

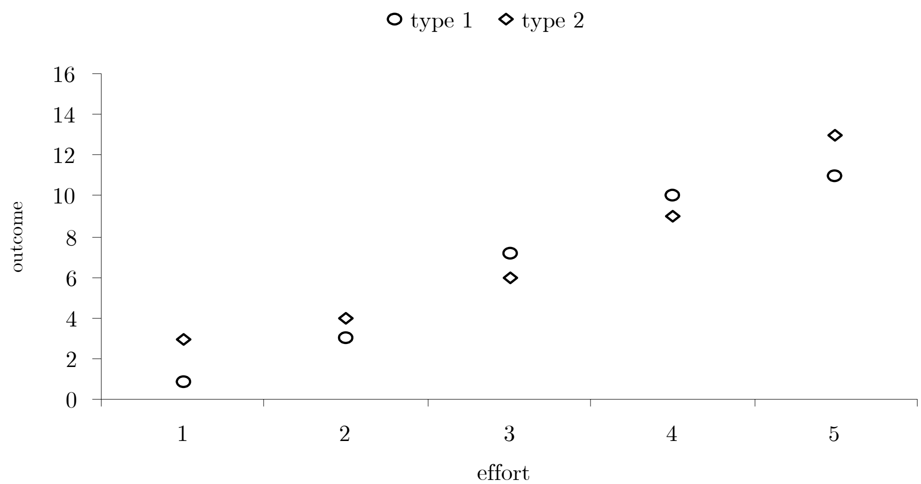

Figure 1: Relation between effort and outcome for two types

Either one tries to equalize the outcomes of different types at the same effort level: this is the CO (*compensating outcomes*) approach to which Roemer's rule belongs. Define the social option x^j which maximizes the minimal outcome of the different types at effort level j, i.e.

$$x^j = \arg \max_{x \in X} \min_{i \in M} U_i^j (x) . \tag{1}$$

It is rather unlikely that x^j will be the same for all effort levels. Therefore, it is necessary to look for an adequate aggregation rule over the different effort levels. Roemer proposes — without much justification— a simple additive specification for this purpose. Roemer's social objective becomes

$$\max_{x \in X} \sum_{j \in N} \min_{i \in M} U_i^j (x) . \tag{2}$$

Figure 2 presents the minimal outcomes for each effort level in black; the Roemer rule wants to maximize the sum of these outcomes.

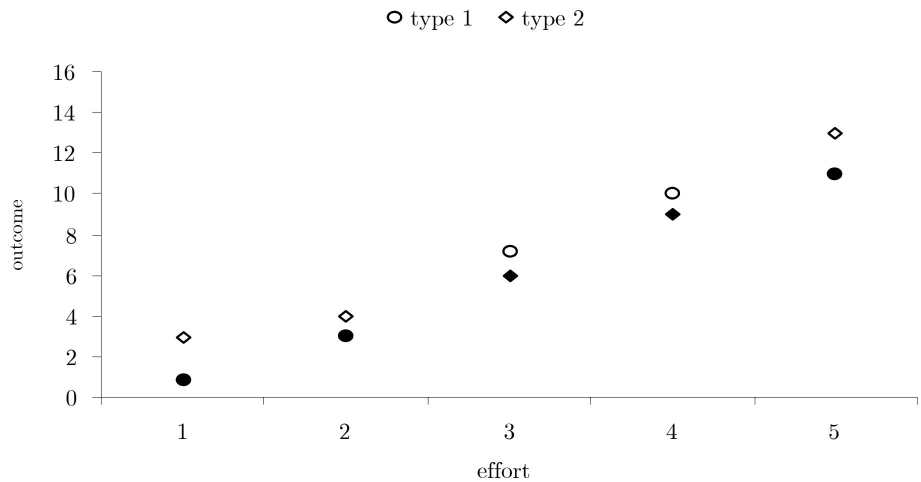

Figure 2: Roemer's sum-of-mins rule

Or one tries to equalize the opportunity sets of the different types; this is in line with the CS (*compensating sets*) approach to which Van de gaer's rule belongs. The Van de gaer rule follows Bossert (1997), who values an opportunity set solely on the basis of the corresponding outcomes. Here again one has to define how to aggregate these outcomes. Again without too much justification, Van de gaer proposes a simple additive specification: the value of the opportunity set of type i under option x becomes

$$\sum_{j\in N} U_i^j(x).$$

Equalizing opportunity sets boils down to equalizing their valuation; the maximin-rule readily suggests itself. Van de gaer's rule maximizes the value of the opportunity set of the most disadvantaged type:

$$\max_{x\in X}\min_{i\in M}\sum_{j\in N} U_i^j(x). \tag{3}$$

Figure 3 shows the outcomes of the most disadvantaged (lowest sum of outcomes) —here type 1— in black; the Van de gaer rule wants to maximize the sum of these outcomes.

Comparing (2) and (3) shows that the only formal difference is the interchange of the min and the sum operator.[3] If the same type is the least advantaged at each effort level, both criteria lead to the same policy. However, although they are formally similar, the basic intuition behind both rules —CO versus CS— is very different.[4] We provide an example where the ranking of two social options is different between both rules.

[3] In fact, Roemer (2002) presents the Van de gaer-rule as an alternative compromise procedure in case the options x^j, defined in equation (1), are different for some effort levels. Roemer has no strong preference for any of these alternatives over the other. In this spirit, our paper can be read as a comparison of different compromise procedures in the Roemer-approach. However, we feel that the differences are deeper than suggested by this interpretation.

[4] Kolm (2001) suggests to interpret Roemer's criterion via the normative concept "desert" and Van de gaer's in terms of its dual, "merit".

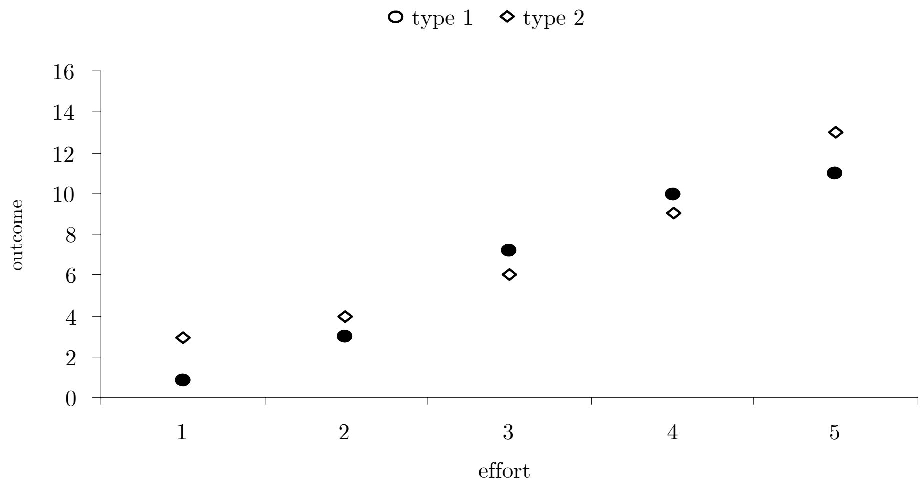

Figure 3: Van de gaer's min-of-sums rule

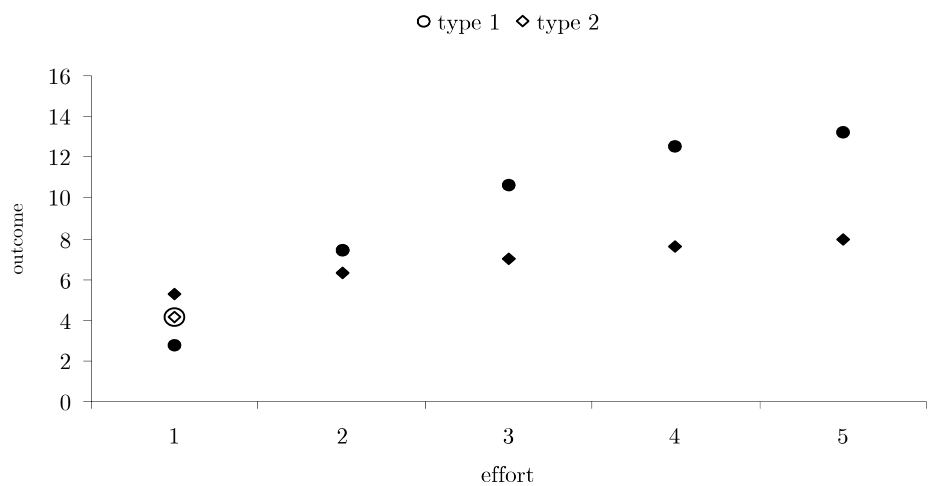

Figure 4: Roemer versus Van de gaer

Figure 4 presents the opportunity sets of two types (bullets and diamonds) over five effort levels in black; except for the lowest effort level, type 1 is everywhere better off. Now, suppose we can redistribute outcomes for the lowest effort level only, such that outcome levels after the transfer (in white) coincide. According to the Roemer rule, this policy is an improvement, although type 1 dominates type 2 after the change; thus, the CO-approach neglects opportunity set considerations. The Van de gaer rule, would prefer the inverse policy; thus, the CS-approach may increase the outcome differences for individuals who exert the same effort.

This example suggests that there is really a basic difference between the Roemer rule and the Van de gaer rule —a difference which goes deeper than a simple normalization. At the same time, however, both rules have analogous weaknesses. Consider a situation with only one type. Both the Roemer rule and the Van de gaer rule will then evaluate different policies in a utilitarian fashion, i.e. as the sum of the outcomes at the different effort levels. It has been argued (see, e.g., Fleurbaey, 1998) that this goes against the intuition of the idea of equal opportunity because the government should not intervene and should simply accept the laissez-faire outcomes, if the differences between outcomes result only from differences in effort.

It is true that this is the usual interpretation of equality of opportunity. At the same time, however, a broader idea of responsibility-sensitivity does *not* necessarily imply an absolute respect for laissez-faire. It is possible to take the position that the social evaluation should focus on opportunities (and opportunity sets) rather than on outcomes while at the same time postulating a definite idea about what should be the optimal form of these sets. In the latter interpretation it seems useful to think about ways to formulate a more flexible approach than the sum-specification of both the Roemer rule and the Van de gaer rule.

The main aim of this paper is to further explore the basic differences between the standard Roemer rule and the Van de gaer rule (section 5), and to look for more flexible extensions (section 6). Sections 3 and 4 introduce the necessary notation.

3 Social orderings

Notation We assume strong neutrality as defined by Roberts (1995) for his "extensive" social choice framework. As a consequence, we can focus on a social ordering[5] R over outcome vectors $u = \left(u_1^1, u_1^2, \ldots, u_m^n\right)$, rather than on an extensive social welfare functional defined over profiles $U = \left(U_1^1, U_1^2, \ldots, U_m^n\right)$. For convenience, we will consider outcome matrices:

$$u = \left[u_i^j\right] = \begin{bmatrix} u_1^1 & & \cdots & & u_1^n \\ \cdots & & & & \cdots \\ u_i^1 & \cdots & u_i^j & \cdots & u_i^n \\ \cdots & & & & \cdots \\ u_m^1 & & \cdots & & u_m^n \end{bmatrix}$$

Row $u_i = \left(u_i^1, \ldots, u_i^n\right)$ represents the outcome vector for type i (the opportunities), while column $u^j = \left(u_1^j, \ldots, u_m^j\right)$ represents the outcome vector for effort level j. We introduce three rank-operators. The first one ranks outcomes within each column of a matrix u in an increasing way; the resulting matrix equals $\widetilde{u}$, with $\widetilde{u}_1^j$ representing the lowest outcome for effort level j in the original matrix u and so on. The second one reranks the rows in an increasing way, on the basis of the sum of their outcomes; the resulting matrix equals $\overset{+}{u}$, with $\overset{+}{u}_1 = \left(\overset{+}{u}{}_1^1, \ldots, \overset{+}{u}{}_1^n\right)$ the opportunities of the type with the lowest sum of outcomes in the original matrix u and so on. The final one also reranks the rows of a strictly positive matrix in an increasing way, but on the basis of the product of its outcomes; $\overset{\times}{u}$ is the matrix obtained from u in this way.

The Roemer rule and the Van de gaer rule We present Roemer and Van de gaer's proposal, slightly modified to satisfy the strong Pareto principle; we therefore replace maximin

[5] An ordering is a complete and transitive binary relation. P and I denote the corresponding asymmetric and symmetric relation.

by leximin. The *Roemer rule* can be rephrased as lexicographically maximizing the sum of the minimal outcomes at each effort level; uRv holds, if and only if[6]

$$\begin{cases} \text{either, } \sum_{j\in N} \widetilde{u}_i^j = \sum_{j\in N} \widetilde{v}_i^j, i = 1, \ldots, m \ , \\ \text{or, } \exists s, 1 \leq s \leq m : \sum_{j\in N} \widetilde{u}_i^j = \sum_{j\in N} \widetilde{v}_i^j, i < s \text{ and } \sum_{j\in N} \widetilde{u}_s^j > \sum_{j\in N} \widetilde{v}_s^j. \end{cases}$$

In the same vein, the *Van de gaer rule* can be reformulated as lexicographically maximizing the minimal sum of outcomes for the different types; uRv holds, if and only if[7]

$$\begin{cases} \text{either, } \sum_{j\in N} \overset{+j}{u_i} = \sum_{j\in N} \overset{+j}{v_i}, i = 1, \ldots, m \ , \\ \text{or, } \exists s, 1 \leq s \leq m : \sum_{j\in N} \overset{+j}{u_i} = \sum_{j\in N} \overset{+j}{v_i}, i < s \text{ and } \sum_{j\in N} \overset{+j}{u_s} > \sum_{j\in N} \overset{+j}{v_s} . \end{cases}$$

The main focus of this paper will be on the characterization of these two basic social orderings. However, as mentioned already in the previous section, the sum-component in both rules has hardly been justified in the literature until now. As a by-product of our characterization we will better understand the consequences of this additive specification. In section 6, we will characterize some alternative rules.

The multiplicative Roemer rule and the Van de gaer rule Instead of adding, we may also multiply outcomes over different effort levels. Restricting the domain to strictly positive outcome matrices, the *product-Roemer rule* ranks uRv, if and only if:

$$\begin{cases} \text{either, } \prod_{j\in N} \widetilde{u}_i^j = \prod_{j\in N} \widetilde{v}_i^j, i = 1, \ldots, m \ , \\ \text{or, } \exists s, 1 \leq s \leq m : \prod_{j\in N} \widetilde{u}_i^j = \prod_{j\in N} \widetilde{v}_i^j, i < s \text{ and } \prod_{j\in N} \widetilde{u}_s^j > \prod_{j\in N} \widetilde{v}_s^j. \end{cases}$$

The *product-Van de gaer rule* ranks uRv, if and only if

$$\begin{cases} \text{either, } \prod_{j\in N} \overset{\times j}{u_i} = \prod_{j\in N} \overset{\times j}{v_i}, i = 1, \ldots, m \ , \\ \text{or, } \exists s, 1 \leq s \leq m : \prod_{j\in N} \overset{\times j}{u_i} = \prod_{j\in N} \overset{\times j}{v_i}, i < s \text{ and } \prod_{j\in N} \overset{\times j}{u_s} > \prod_{j\in N} \overset{\times j}{v_s} . \end{cases}$$

Flexible Roemer rule and Van de gaer rule Finally, we introduce a Kolm-Atkinson-Sen (KAS) and Kolm-Pollak (KP) specification, as suggested by Kolm (2001). Let $g : \mathbb{R}^m_{++} \to \mathbb{R}_{++}$ and $h : \mathbb{R}^n_{++} \to \mathbb{R}_{++}$ be strictly increasing, symmetric, separable and homogeneous functions, and define $\exp\left(u^j\right) = \left(\exp u_1^j, \ldots, \exp u_m^j\right)$ and $\exp\left(u_i\right) = \left(\exp u_i^1, \ldots, \exp u_i^n\right)$. The *KAS-Roemer family* and the *KP-Roemer family* contain social orderings preferring u to v, if and only if respectively[8]

$$\left(\sum_{j\in N} \left(g\left(u^j\right)\right)^\rho\right)^{\frac{1}{\rho}} \geq \left(\sum_{j\in N} \left(g\left(v^j\right)\right)^\rho\right)^{\frac{1}{\rho}} \text{ and } \left(\sum_{j\in N} \left(g\left(\exp\left(u^j\right)\right)\right)^\rho\right)^{\frac{1}{\rho}} \geq \left(\sum_{j\in N} \left(g\left(\exp\left(v^j\right)\right)\right)^\rho\right)^{\frac{1}{\rho}} .$$

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[6] If one allows for a different number n_{ij} of individuals with characteristics $ij \in MN$, the mins have to be weighted by proportions $p_j = \frac{\sum_i n_{ij}}{\sum_{ij} n_{ij}}$.

[7] If one allows for a different number n_{ij} of individuals with characteristics $ij \in MN$, Van de gaer's rule has to be based on sums of outcomes, weighted by the conditional proportion $p_{ij/i} = \frac{n_{ij}}{\sum_j n_{ij}}$.

[8] As usual, when $\rho = 0$ the KAS-specification is equal to a log-utilitarian rule.

The *KAS-Van de gaer family* and *KP-Van de gaer family* invert the aggregation steps. They contain social orderings preferring u to v, if and only if respectively:

$$\left(\sum_{i\in M}\left(h\left(u_i\right)\right)^{\rho}\right)^{\frac{1}{\rho}} \geq \left(\sum_{i\in M}\left(h\left(v_i\right)\right)^{\rho}\right)^{\frac{1}{\rho}} \text{ and } \left(\sum_{i\in M}\left(h\left(\exp\left(u_i\right)\right)\right)^{\rho}\right)^{\frac{1}{\rho}} \geq \left(\sum_{i\in M}\left(h\left(\exp\left(v_i\right)\right)\right)^{\rho}\right)^{\frac{1}{\rho}}.$$

4 Properties

We focus on a social ordering defined over matrices in a domain $\mathcal{D}$, with $\mathcal{D}$ equal to $\mathbb{R}^{m\times n}$ or $\mathbb{R}^{m\times n}_{++}$:

- **UD** *(unrestricted domain).*R is defined on $\mathcal{D}=\mathbb{R}^{m\times n}$.
- **PD** *(positive domain).* R is defined on $\mathcal{D}=\mathbb{R}^{m\times n}_{++}$.

Continuity is defined as:

- **CON** *(continuity).*
  For each $u\in\mathcal{D}$; the better-than and worse-than sets $\{v\in\mathcal{D}\,|vRu\}$ and $\{v\in\mathcal{D}\,|uRv\}$ are closed.

Efficiency boils down to the strong Pareto principle, defined as:

- **SP** *(strong Pareto).*
  For all $u,v\in\mathcal{D}$; if $u_i^j\geq v_i^j$ for all $ij\in MN$, then uRv. If, in addition, there exists a $kl\in MN$ such that $u_k^l>v_k^l$, then uPv.

The main differences between the CO-approach and the CS-approach are linked to separability, impartiality and compensation. The columns in Table 1 present the different axioms within these three classes, while the rows indicate the two main approaches. The symbol $+$ (resp. $-$) indicates whether an axiom is typical (resp. atypical) for one of both approaches; the symbol $\bullet$ refers to axioms that will be used in the next section to characterize the Roemer rule and the Van de gaer rule.

Table 1: **Compensating outcomes versus compensating sets**

	separability (SE) between effort (N)		type (M)		impartiality (SI) for effort (N)		type (M)		compensation (C) (for type only)	
	SE_N^*	SE_N	SE_M^*	SE_M	SI_N^*	$\widetilde{\mathrm{SI}}_N$	SI_M^*	SI_M	C	DC
CO	$+$	$+$	$-$	$+$	$-$	$\bullet$	$\bullet$	$+$	$+$	$\bullet$
CS	$-$	$+$	$+$	$+$	$\bullet$	$+$	$-$	$\bullet$	$-$	$\bullet$

Separability For reasons that will become clear later on, separability assumptions do not play a role in the characterization of the basic Roemer rule and the Van de gaer rule (see table 1). They are more important to characterize the flexible extensions in section 6. In a CS-approach it seems straightforward to impose a separability requirement between the different rows (opportunities of a type). Strong separability between types eliminates indifferent types, i.e. types with identical opportunities under two social options do not play a role:

- **SE**$^*_M$ *(strong separability between types).*
  For all $u, v, u', v' \in \mathcal{D}$ and for all subsets $G \subseteq M$; if $u_i^j = v_i^j$ and $u_i^{'j} = v_i^{'j}$ for all $ij \in GN$, while $u_i^j = u_i^{'j}$ and $v_i^j = v_i^{'j}$ for all $ij \in MN \backslash GN$, then uRv if and only if $u'Rv'$.

In the same way we can define strong separability between effort (columns). This axiom eliminates indifferent effort levels, i.e. effort levels with identical outcome vectors. This axiom is very intuitive for the CO approach.

- **SE**$^*_N$ *(strong separability between effort).*
  For all $u, v, u', v' \in \mathcal{D}$ and for all subsets $G \subseteq N$; if $u_i^j = v_i^j$ and $u_i^{'j} = v_i^{'j}$ for all $ij \in MG$, while $u_i^j = u_i^{'j}$ and $v_i^j = v_i^{'j}$ for all $ij \in MN \backslash MG$, then uRv if and only if $u'Rv'$.

For both separability requirements, we can also introduce a weaker version, which requires separability in case of specific subdomains. $\mathcal{D}_M$ contains matrices where outcomes in each row are constant: effort has no effect on the outcomes of each type. $\mathcal{D}_N$ consists of matrices where outcomes in each column are constant: type has no effect on the outcomes for each effort level.

- **SE**$_M$ *(separability between types).*
  SE$^*_M$ holds for all $u, v, u', v' \in \mathcal{D}_M$.
- **SE**$_N$ *(separability between effort).*
  SE$^*_N$ holds for all $u, v, u', v' \in \mathcal{D}_N$.

Impartiality Much more important are the impartiality requirements. Suppes indifference implies that the "names" of the types (resp. effort levels) do not matter. We will define it in two different strengths: we can permute rows (resp. columns), but stronger, we can also permute outcomes within each column (resp. row) separately. Let σ denote a permutation of a set. When reasoning within a CS-approach, the following combination of axioms seems reasonable:

- **SI**$_M$ *(Suppes indifference for types).*
  For all $u, v \in \mathcal{D}$ and for each permutation σ on M; if $v = \left[u^j_{\sigma(i)}\right]$, then uIv.
- **SI**$^*_N$ *(strong Suppes indifference for effort).*
  For all $u, v \in \mathcal{D}$ and for each m-tuple $(\sigma_1, \ldots, \sigma_m)$ of permutations on N; if $v = \left[u_i^{\sigma_i(j)}\right]$, then uIv.

The first axiom SI$_M$ is a natural requirement from a CS perspective. It imposes that the different types should be treated in a symmetric way: their opportunity vectors can be permuted without influencing the social evaluation. The second one, SI$^*_N$, is also a plausible requirement if we look at the rows of the matrices as embodying the values of the opportunity sets to which different individuals have access. It imposes that only the whole vector of opportunities matters, not the specific link between each effort level and the resulting outcome.

On the other hand, in the CO-approach we can reasonably assume:

- $\mathbf{SI}^*_M$ *(strong Suppes indifference for types).*
  For all $u, v \in \mathcal{D}$ and for each n-tuple $\left(\sigma^1, \ldots, \sigma^n\right)$ of permutations on M;if $v = \left[u^j_{\sigma^j(i)}\right]$, then uIv.

- $\mathbf{SI}_N$ *(Suppes indifference for effort).*
  For all $u, v \in \mathcal{D}$ and for each permutation σ on N; if $v = \left[u_i^{\sigma(j)}\right]$, then uIv.

In the CO-setting, SI^*_M is a very natural requirement. When comparing the outcomes for different types at the same effort level, the identity of the types should not matter. SI_N permutes entire columns, or the names of the effort levels do not matter. We could also consider a version of Suppes indifference in between SI_N and SI^*_N. Recall the rank operator $\sim$; we define

$$\widetilde{\mathcal{D}} = \{u \in \mathcal{D} \,|\, \exists v \in \mathcal{D} \text{ such that } u = \widetilde{v}\}.$$

Notice that the Roemer rule and the Van de gaer rule coincide on $\widetilde{\mathcal{D}}$. Therefore, both rules satisfy the following version of Suppes indifference:

- $\widetilde{\mathbf{SI}}_N$ *(ordered Suppes indifference for effort).*
  SI^*_N holds for all $u, v \in \widetilde{\mathcal{D}}$.

Compensation In the CO approach, compensation requires equalization of the outcomes of the different types who exerted the same effort. In the spirit of Hammond (1976), we define:

- **C** *(compensation).*
  For all $u, v \in \mathcal{D}$ and for all $ij, kj \in MN$; if $v^j_k > u^j_k \geq u^j_i > v^j_i$, while $u^h_g = v^h_g$ for all other $gh \neq ij, kj$, then uRv.

In the CS-approach however, compensation focuses on the equalization of opportunity vectors. Valuing opportunity vectors is a difficult problem and we do not want to impose strong assumptions about it at this stage. We therefore start from the simple idea that an opportunity set is "better" than another if it gives rise to higher outcomes for all effort levels. Given two types i and k, we write $u_i \geq u_k$ to denote $u^j_i \geq u^j_k$, for all j in N; in this case, we say that type i dominates type k. Adding such a dominance requirement to the previous compensation axiom gives us dominance compensation:[9]

- **DC** *(dominance compensation).*
  For all $u, v \in \mathcal{D}$ and for all $ij, kj \in MN$; if $v_i \geq v_k$ and $v^j_i > u^j_i \geq u^j_k > v^j_k$, while $u^h_g = v^h_g$ for all other $gh \neq ij, kj$, then uRv.

The Van de gaer rule implies much more than only a dominance condition with respect to the evaluation of opportunity sets. It incorporates the following utilitarian compensation principle:[10]

[9] This differs from the approach taken by Ok and Kranich (1998) who advocate to define equalization of opportunity sets for a given "cardinality-based" ordering of all possible opportunity sets. Our "ordinality-based" notion of dominance compensation is the equivalent of "opportunity dominance for a policy" as introduced by Hild and Voorhoeve (2001).

[10] The cartesian product $\{i\} \times N$ is abbreviated as iN.

- **UC** *(utilitarian compensation).*
  For all $u, v \in \mathcal{D}$ and for all types $i, k \in M$; if $\sum_{j\in N} v_k^j > \sum_{j\in N} u_k^j \geq \sum_{j\in N} u_i^j > \sum_{j\in N} v_i^j$, while $u_g^h = v_g^h$ for all $gh \notin iN \cup kN$, then uRv.

We do not use this stronger axiom for the characterization of the Van de gaer rule, but we show that it is implied by the combination of other axioms. For later reference, we finish by introducing the idea of utilitarian neutrality. If the sum of opportunities remains the same for all types, then we should rank the matrices as indifferent:

- **UN** *(utilitarian neutrality).*
  For all $u, v \in \mathcal{D}$; if $\sum_{j\in N} u_i^j = \sum_{j\in N} v_i^j$ for all i in M, then uIv.

5 Results

The Roemer rule and the Van de gaer rule Roemer's CO-approach focuses on the differences between the outcomes for different types at the same effort level, i.e. on the inequality within the columns of the outcome matrix. It then seems natural to require that the social evaluation should be invariant if all outcomes within the columns are changed by the same absolute number or are changed in the same proportion since such changes do not affect absolute or relative inequality, respectively. We will see that the Roemer rule focuses on absolute inequality within a column. Roemer remains agnostic, however, about the differences between the different columns. Therefore, the social evaluation does not depend on the differences between outcomes at different effort levels.

To formalize this idea, recall $\mathcal{D}_N$, the domain of matrices with all elements equal in a column. We define the following axiom:[11]

- $\mathbf{I}_{\mathrm{tf}}^{\mathrm{tn}}$ *(invariance axiom).*
  For all $u, v \in \mathcal{D}$ and for all $\alpha \in \mathcal{D}_N$; uRv if and only if $(u + \alpha)\, R\, (v + \alpha)$.

Notice that $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ implies separability between effort levels SE_N. The axiom $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ means that the social judgment does not change provided that the absolute differences in outcomes at each effort level, i.e. within each column, are preserved. At the same time, since the elements of matrix α may differ over different effort levels, differences in outcomes between the columns are irrelevant for the social evaluation. Within the CO-approach, the $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ axiom forces us to focus on the inequality of outcomes, irrespective of the level at which these inequalities are evaluated.

The $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ axiom is equally relevant in the CS-approach. Indeed, if the difference in outcomes for each effort level remains the same, then the difference in the valuation of the opportunity sets also remains the same. If one wants to focus on the inequality of the sets, the α-transformation (defined in $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$) should therefore not lead to a change in the relative evaluation of different opportunity sets.

We are now ready to formulate our two basic characterization results:[12]

[11] From a formal point of view, the invariance axiom $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ is identical to a two-dimensional invariance assumption, introduced in Ooghe and Lauwers (2003). It amounts to translation-scale measurability (T) combined with full comparability (F) between the different types (for a given effort level) and translation-scale measurability (T) and no comparability (N) between the different effort levels (for a given type).

[12] All the proofs are put together in the appendix.

Proposition 1. For each row in the table, there is only one social ordering R which satisfies the axioms indicated by •; it is the rule in the first column. In addition, + (resp. −) means that the rule satisfies (violates) the corresponding axiom:

	UD	$\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$	SP	DC	SI_M^*	SI_M	SI_N^*	$\widetilde{\mathrm{SI}}_N$
Roemer	•	•	•	•	•	+	−	•
Van de gaer	•	•	•	•	−	•	•	+

Discussion In the proof of proposition 1 (see appendix), we show that all axioms are independent. The separability axioms play no role in this characterization. Their potential role is taken over by the invariance axiom $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ and the equity axiom DC.[13] More surprising is the fact that we use in both cases the same dominance compensation axiom DC. The only substantial differences are to be found in the impartiality axioms SI_M^* and SI_N^*. As we show next, they have strong implications.

The proof of proposition 1 for the Van de gaer rule makes use of the following lemma 1, showing that the combination of the invariance axiom $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ with strong Suppes indifference between effort levels SI_N^* implies utilitarian neutrality UN:

Lemma 1. If a social ordering R defined on $\mathcal{D}$ satisfies UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ and SI_N^*, then it also satisfies UN.

It is clear that UN makes sense only in the CS approach, where sets are measured by the sum of the outcomes of those that have the same compensation characteristic. Lemma 1 shows that, in combination with UD and $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, the axiom SI_N^* brings us already a long way in the direction of the Van de gaer rule. This becomes even more transparent when we impose also the dominance compensation axiom DC, because one can show

Lemma 2. If a social ordering R defined on $\mathcal{D}$ satisfies UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, SI_N^* and DC, then it also satisfies UC.

The other impartiality requirement SI_M^* plays an equally important role in the characterization of the Roemer rule. Imposing it indeed implies a considerable strengthening of the rather weak idea of dominance compensation. This is summarized in the following lemma:

Lemma 3. If a social ordering R defined on $\mathcal{D}$ satisfies UD, SI_M^* and DC, then it also satisfies C.

Note that we described in the previous section how compensation C captures the main intuition of the CO-approach. The basic incompatibility between the Roemer rule and the Van de gaer rule can be summarized in the following lemma:

Lemma 4. Suppose $m, n \geq 2$. A social ordering R defined on $\mathcal{D}$ cannot satisfy UD, SP, C and UN simultaneously.

Combining these various lemmas, we obtain the following message. If we want to keep UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, SP, and DC, we will have to choose either SI_M^*, or SI_N^*. Choosing SI_M^* implies choosing C and choosing SI_N^* implies choosing UN, and even UC. The choice between both impartiality axioms can therefore be seen as the choice between compensating outcomes via C, or compensating

[13] Ooghe and Lauwers (2003) provide an alternative characterization of the Roemer-rule, via (additional) separability axioms, combined with a minimal equity requirement.

opportunity sets via UC, where the value of the opportunity sets is measured as the sum of the outcomes achieved by individuals with the same type. This dichotomy is also the driving force behind the example given in section 2 (figure 4).

6 Extensions

The multiplicative extensions In the previous section, we have shown that both the Roemer rule and the Van de gaer rule satisfy the invariance axiom $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$. It is not obvious that this is an attractive idea. First, it implies an absolute approach to the evaluation of inequality of opportunity. Such an absolute approach is not the most popular in the literature on inequality measurement. Second, it implies a complete agnosticism about the ethical evaluation of differences between the outcomes related to different effort levels.

The most obvious alternative to $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ is to follow the dominant *relative* approach to inequality measurement: within a column we look at relative, rather than absolute differences in outcomes. This means that we only allow for *ratios* of outcomes between types with the same effort to be relevant. Consider Π the domain of diagonal matrices $\beta \in \mathbb{R}^{n\times n}$ with strictly positive diagonal entries $\beta_i^i > 0$. We then propose:

- $\mathbf{I}_{\mathrm{rf}}^{\mathrm{rn}}$ *(invariance axiom)*

  For all $u, v \in \mathcal{D}$ and for all $\beta \in \Pi$; uRv if and only if $(u\beta)\, R\, (v\beta)$.

Similar to $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, this invariance axiom $\mathrm{I}_{\mathrm{rf}}^{\mathrm{rn}}$ implies separability between effort levels SE_N. Furthermore, $\mathrm{I}_{\mathrm{rf}}^{\mathrm{rn}}$ preserves the relative inequality between outcomes for a given effort level. Since this holds for all effort levels, it can be maintained that this does not change the inequality of opportunity sets either, if one takes a relative perspective on inequality. With the necessary change in domain, we get:

Proposition 2. For each row in the table, there is only one social ordering R which satisfies the axioms indicated by •; it is the rule in the first column. In addition, + (resp. −) means that the rule satisfies (violates) the corresponding axiom:

	PD	$\mathrm{I}_{\mathrm{rf}}^{\mathrm{rn}}$	SP	DC	SI_M^*	SI_M	SI_N^*	$\widetilde{\mathrm{SI}}_N$
product-Roemer	•	•	•	•	•	+	−	•
product-Van de gaer	•	•	•	•	−	•	•	+

The flexible extensions We can go further and drop the assumption of non-comparability between columns. Again, this is easily done by exploiting the analogy with the classical invariance requirements in the social choice literature. Define $\mathbf{1}_m^n \in \mathbb{R}^{m\times n}$, a matrix consisting of ones:

- $\mathbf{I}_{\mathrm{tf}}^{\mathrm{tf}}$ *(invariance axiom).*

  For all $u, v \in \mathcal{D}$ and for all $a \in \mathbb{R}$; uRv if and only if $(u + a\mathbf{1}_m^n)\, R\, (v + a\mathbf{1}_m^n)$.

- $\mathbf{I}_{\mathrm{rf}}^{\mathrm{rf}}$ *(invariance axiom).*

  For all $u, v \in \mathcal{D}$ and for all $b \in \mathbb{R}_{++}$; uRv if and only if $(bu)\, R\, (bv)$.

$\mathrm{I}^{\mathrm{tf}}_{\mathrm{tf}}$ (resp. $\mathrm{I}^{\mathrm{rf}}_{\mathrm{rf}}$) does not imply separability between effort levels; separability axioms will be imposed directly. Furthermore, $\mathrm{I}^{\mathrm{tf}}_{\mathrm{tf}}$ (resp. $\mathrm{I}^{\mathrm{rf}}_{\mathrm{rf}}$) only compares situations that keep the absolute (resp. relative) differences between outcomes at all effort levels constant. These weaker axioms allow us to express a wide spectrum of opinions about inequalities that are due to type and effort. The following proposition characterizes the Kolm-Atkinson-Sen (KAS) and Kolm-Pollak (KP) specification for the Roemer rule and the Van de gaer rule:

Proposition 3. For each row in the table, there is only one family of social orderings R which satisfies the axioms indicated by •; it is the family in the first column. In addition, + (resp. −) means that *each* rule in the family satisfies (violates) the corresponding axiom:

	PD	$\mathrm{I}^{\mathrm{rf}}_{\mathrm{rf}}$	SP	CON	SE^*_M	SE_M	SE^*_N	SE_N	SI^*_M	SI_M	SI^*_N	SI_N
KAS-R	•	•	•	•		•	•	+	•	+		•
KAS-VDG	•	•	•	•	•	+		•		•	•	+
	UD	$\mathrm{I}^{\mathrm{tf}}_{\mathrm{tf}}$										
KP-R	•	•	•	•		•	•	+	•	+		•
KP-VDG	•	•	•	•	•	+		•		•	•	+

R stands for Roemer and VDG for Van de gaer.

It remains true that the main differences between the (broad) Roemer rule and (broad) Van de gaer rule (and therefore also between the CO and the CS approaches to equality of opportunity) are revealed by the impartiality axioms, but they are here reinforced by specific separability axioms.

We can conclude that the invariance axiom $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$ is crucial in explaining the sum-assumption which is imposed both by Roemer and Van de gaer. Alternatives to these rules are readily available for those who do not like the (strong) implications of $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$: absolute inequality measurement within a column, i.e. for a given effort level, and complete neglect of differences in outcome levels between columns, i.e. for different effort levels. Of course, none of these alternatives counters the criticism that with one type the laissez-faire outcomes should be respected. However, if one accepts the broader interpretation of responsibility-sensitivity in which the social evaluation also depends on the form of the opportunity sets, the increased flexibility gained by relaxing $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$ seems a definite improvement. Note that by imposing even weaker invariance requirements than $\mathrm{I}^{\mathrm{tf}}_{\mathrm{tf}}$ and $\mathrm{I}^{\mathrm{rf}}_{\mathrm{rf}}$, many additional functional forms become possible.

7 Conclusion

Although the Roemer rule and the Van de gaer rule are formally similar, their basic intuition is very different. This difference goes deeper than a mere difference in normalization. The former exploits the idea of compensating for different outcomes, if these differences follow from differences in non-responsibility or compensation factors. The latter focuses on the evaluation of opportunity sets and wants to make their "value" as equal as possible. As we have shown, both approaches are incompatible in general. While they lead to similar policy prescriptions in many cases (and to identical policy prescriptions if there is one type whose opportunity set is dominated by all the other types), this should not detract us from the differences in ethical inspiration. It would be interesting to further explore these differences in concrete applications.[14] More empirical work could be very fruitful in this regard.

Both rules satisfy an invariance axiom which we have labeled $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$. It is reflected in their additive specification. The axiom $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ imposes that the social evaluation does not change if the same constant is added to the outcomes of all types at a given effort level. This essentially implies an absolute approach to inequality measurement. Moreover, these constants may differ at different effort levels, implying that differences between the outcomes related to different effort levels do not matter. All these assumptions are debatable. However, we have shown that introducing other invariance axioms leads to rules of the Roemer family and the Van de gaer family with a more flexible specification.

The importance of this increased flexibility should not be underestimated. As has been noted by Fleurbaey (1998), neither the Roemer rule nor the Van de gaer rule satisfy the traditional notion of equality of opportunity implying that the laissez-faire outcome should be respected in cases where there is only one type. This is one specific notion of responsibility-sensitive egalitarianism, however. It can also be argued that responsibility matters, i.e. that effort should be rewarded, while at the same time putting forward definite ideas about *how* it should be rewarded and, more specifically, without accepting that the laissez-faire rewards to effort are necessarily ideal from an ethical point of view. This requires that the form of opportunity sets is evaluated, even if there is only one type. The additive (or utilitarian) specification underlying the original Roemer rule and the Van de gaer rule may not be the most attractive candidate for such an evaluation exercise. The more flexible specifications introducing a parameter of inequality aversion can then be seen as a first step into the direction of a broader notion of responsibility-sensitive egalitarianism.

[14] Van de gaer et al. (2001) explore the justification of the rules in the context of the measurement of intergenerational mobility. Schokkaert et al. (2002) calculate the optimal linear income tax for these rules.

Appendix

Proof of lemmas

Proof of lemma 1 If a social ordering R defined on $\mathcal{D}$ satisfies UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ and SI_N^*, then it also satisfies UN. Consider a matrix $u \in \mathcal{D}$ and two couples ij, ik in MN. Construct a matrix v by permuting elements ij and ik in matrix u. Using SI_N^* we have uIv. Add an arbitrary amount $a \in \mathbb{R}$ to the outcomes in the j-th column of both matrices u and v to get $u^{'}$ and $v^{'}$; using $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, we have $u^{'}Iv^{'}$. Permute the elements ij and ik in matrix $u^{'}$ to get $u^{''}$ and by SI_N^* and transitivity of R, we have $u^{''}Iv^{'}$. Subtract the same amount a from the elements in the j-th column of both matrices $u^{''}$ and $v^{'}$ to get $u^{'''}$ and $v^{''} = v$; from $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, we have $u^{'''}Iv$. By construction, the matrix $u^{'''}$ is everywhere equal to v, except for outcomes $u_i^{'''j} = v_i^j - a$ and $u_i^{'''k} = v_i^k + a$. Hence, we may remove outcome mass within a row of a matrix v, without changing social welfare, more precisely:

- Transfer principle.

  For all $v \in \mathcal{D}$, for all $ij, ik \in MN$ and for all $a \in \mathbb{R}$; uIv, with $u_i^j = v_i^j + a$, $u_i^k = v_i^k - a$ and $u_g^h = v_g^h$ for all $gh \in MN \setminus \{ij, ik\}$.

On the basis of this "transfer principle" (used repeatedly, if necessary), UN must hold, as required. □

Proof of lemma 2 If a social ordering R defined on $\mathcal{D}$ satisfies UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, SI_N^* and DC, then it also satisfies UC. From lemma 1, we know that UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$ and SI_N^* implies UN. We prove that adding DC also implies UC. Suppose the antecedents of UC are true for two matrices $u, v \in \mathcal{D}$ and two types i, k in M, i.e. $\sum_{j\in N} v_k^j > \sum_{j\in N} u_k^j \geq \sum_{j\in N} u_i^j > \sum_{j\in N} v_i^j$, while $u_g^h = v_g^h$ for all $gh \notin G = iN \cup kN$. Construct two matrices $u^{'}$ and $v^{'}$ as follows:

$$(i)\ v_k^{'1} = \textstyle\sum_{j\in N} v_k^j \quad (ii)\ u_k^{'1} = \sum_{j\in N} u_k^j \quad (iii)\ u_i^{'1} = \sum_{j\in N} u_i^j \quad (iv)\ v_i^{'1} = \sum_{j\in N} v_i^j$$
$$(v)\ u_g^{'h} = v_g^{'h} = 0, \forall gh \neq i1, k1\ \&\ gh \in G \quad (vi)\ u_g^{'h} = u_g^h\ \&\ v_g^{'h} = v_g^h, \forall gh \notin G.$$

From UN, we have $uIu^{'}$ and $v^{'}Iv$. But by construction, DC applies and we have $u^{'}Rv^{'}$. From transitivity also uRv holds. □

Proof of lemma 3 If a social ordering R defined on $\mathcal{D}$ satisfies UD, SI_M^* and DC, then it also satisfies C. Suppose the antecedents of C are true for $u, v \in \mathcal{D}$ and for ij, kj in MN, i.e. $v_k^j > u_k^j \geq u_i^j > v_i^j$, while $u_g^h = v_g^h$ for all other $gh \neq ij, kj$. Construct two matrices $u^{'}$ and $v^{'}$ by reranking all columns different from j, either increasingly or decreasingly depending on whether $i < k$ or $i > k$. From SI_M^*, we have $uIu^{'}$ and $vIv^{'}$. By construction, we can apply DC to get $u^{'}Rv^{'}$ and thus also uRv must hold, as required. □

Proof of lemma 4 Suppose $m, n \geq 2$. A social ordering R defined on $\mathcal{D}$ cannot satisfy UD, SP, C and UN simultaneously. First, consider the following four matrices

$$u = \begin{bmatrix} 3 & 8 \\ 3 & 6 \end{bmatrix}, v = \begin{bmatrix} 2 & 8 \\ 4 & 6 \end{bmatrix}, u^{'} = \begin{bmatrix} 2 & 10 \\ 4 & 5 \end{bmatrix}, v^{'} = \begin{bmatrix} 3 & 9 \\ 3 & 6 \end{bmatrix}$$

Using C, we have uRv and $vRu^{'}$. Using UN, we have $u^{'}Iv^{'}$. Everything together, we obtain by transitivity that $uRv^{'}$ must hold, which contradicts SP. We can always embed these four matrices in larger ones belonging to $\mathbb{R}^{m\times n}$, which completes the proof.[15] □

Proof of proposition 1

A. Characterization of the Roemer rule It is easy to verify existence: the Roemer rule satisfies all axioms UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, SP, DC, SI_M^* and $\widetilde{\mathrm{SI}}_N$. To prove uniqueness, we need the following lemma, which allows us to use a transfer principle, as in lemma 1, but for ordered matrices:

Lemma 5. If a social ordering R defined on $\mathcal{D}$ satisfies UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, and $\widetilde{\mathrm{SI}}_N$, then it also satisfies the following "ordered" transfer principle: for all $v \in \widetilde{\mathcal{D}}$, for all $ij, ik \in MN$ such that $v_{i-1}^j = v_{i-1}^k$ (if $i \neq 1$) and for all a with (if $i \neq 1$) $v_{i-1}^j - v_i^j \leq a \leq v_i^k - v_{i-1}^k$; uIv holds, with (i) $u_l^j = v_l^j + a$ and $u_l^k = v_l^k - a$, for all $l \in G = \{i, i+1, \ldots, m\}$ (ii) $u_g^h = v_g^h$ for all $gh \in MN \setminus (Gj \cup Gk)$.[16]

Proof. Consider a matrix $v \in \widetilde{\mathcal{D}}$, $ij, ik \in MN$ such that $v_{i-1}^j = v_{i-1}^k$ (if $i \neq 1$) and a with $v_{i-1}^j - v_i^j \leq a \leq v_i^k - v_{i-1}^k$. Suppose $j < k$; the other case is analogous. Consider the following sequence of transformations on v (in vector notation); an explanation follows:

$$\begin{aligned}
v &= \left(v^1; \ldots; \underbrace{v_1^j, \ldots, v_i^j, \ldots, v_m^j}_{v^j}; \ldots; \underbrace{v_1^k, \ldots, v_i^k, \ldots, v_m^k}_{v^k}; \ldots; v^n\right), \\
v^{'} &= \left(v^1; \ldots; v_1^j + a, \ldots, v_i^j + a, \ldots, v_m^j + a; \ldots; v_1^k, \ldots, v_i^k, \ldots, v_m^k; \ldots; v^n\right), \\
v^{''} &= \left(v^1; \ldots; v_1^j + a, \ldots, v_{i-1}^j + a, v_i^k, \ldots, v_m^k; \ldots; v_1^k, \ldots, v_{i-1}^k, v_i^j + a, \ldots, v_m^j + a; \ldots; v^n\right), \\
v^{'''} &= \left(v^1; \ldots; v_1^j, \ldots, v_{i-1}^j, v_i^k - a, \ldots, v_m^k - a; \ldots; v_1^k, \ldots, v_{i-1}^k, v_i^j + a, \ldots, v_m^j + a; \ldots; v^n\right), \\
v^{''''} &= \left(v^1; \ldots; v_1^j, \ldots, v_{i-1}^j, v_i^j + a, \ldots, v_m^j + a; \ldots; v_1^k, \ldots, v_{i-1}^k, v_i^k - a, \ldots, v_m^k - a; \ldots; v^n\right).
\end{aligned}$$

$v^{'}$ is derived from v by adding a to column j, $v^{''}$ is derived from $v^{'}$, by permuting elements ij and ik for all i in G (notice that the constraint for a ensures that $v^{''} \in \widetilde{\mathcal{D}}$), $v^{'''}$ is derived from $v^{''}$ by subtracting a from column j, and $v^{''''}$ is derived from $v^{'''}$ by permuting again elements ij and ik for all i in G (and again, $v^{''''} \in \widetilde{\mathcal{D}}$). We have $v^{''}Rv^{'} \Leftrightarrow v^{'''}Rv$ and $v^{'}Rv^{''} \Leftrightarrow vRv^{'''}$ (via $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$), and $v^{''}Iv^{'}$ and $v^{''''}Iv^{'''}$ (via $\widetilde{\mathrm{SI}}_N$). Combining both, we have $v^{'}Rv^{'} \Leftrightarrow v^{''''}Rv$ and $v^{'}Rv^{'} \Leftrightarrow vRv^{''''}$, and due to reflexivity, we must have $v^{''''}Iv$. It suffices to see that $u = v^{''''}$, and thus uIv, as required. □

Proof of proposition 1A. Consider an ordering R on $\mathcal{D}$ which satisfies UD, $\mathrm{I}_{\mathrm{tf}}^{\mathrm{tn}}$, SP, DC, SI_M^* and $\widetilde{\mathrm{SI}}_N$.

First, we show, for all $a, b \in \mathcal{D}$, that

$$aRb \Leftrightarrow \begin{bmatrix} \frac{1}{n}\sum_{j\in N}\widetilde{a}_1^j & \ldots & \frac{1}{n}\sum_{j\in N}\widetilde{a}_1^j \\ \ldots & & \ldots \\ \frac{1}{n}\sum_{j\in N}\widetilde{a}_m^j & \ldots & \frac{1}{n}\sum_{j\in N}\widetilde{a}_m^j \end{bmatrix} R \begin{bmatrix} \frac{1}{n}\sum_{j\in N}\widetilde{b}_1^j & \ldots & \frac{1}{n}\sum_{j\in N}\widetilde{b}_1^j \\ \ldots & & \ldots \\ \frac{1}{n}\sum_{j\in N}\widetilde{b}_m^j & \ldots & \frac{1}{n}\sum_{j\in N}\widetilde{b}_m^j \end{bmatrix}.$$

[15] This result also holds (i) on the positive domain, and/or (ii) without completeness of the social relation.
[16] $Gj = G \times \{j\}$.

Consider $a, b \in \mathcal{D}$ such that aRb and consider the following steps:

- Recall the rerank operator $\sim$. Define $c, d \in \widetilde{\mathcal{D}}$ with $c = \widetilde{a}$ and $d = \widetilde{b}$; using SI^*_M we have cRd.
- Use the ordered transfer principle described in lemma 5 (repeatedly, if necessary) to obtain matrices e, f from c, d with

$$e_1 = \left(\frac{1}{n} \sum_{j \in N} c_1^j, \dots, \frac{1}{n} \sum_{j \in N} c_1^j \right); f_1 = \left(\frac{1}{n} \sum_{j \in N} d_1^j, \dots, \frac{1}{n} \sum_{j \in N} d_1^j \right).$$

  From lemma 5 and transitivity, we have eRf.
- Use the ordered transfer principle described in lemma 5 (repeatedly, if necessary) —in a way which leaves the first row unchanged— to obtain matrices g, h from e, f with

$$g_1 = e_1 \text{ and } g_2 = \left(\frac{1}{n} \sum_{j \in N} e_2^j, \dots, \frac{1}{n} \sum_{j \in N} e_2^j \right); h_1 = f_1 \text{ and } h_2 = \left(\frac{1}{n} \sum_{j \in N} f_2^j, \dots, \frac{1}{n} \sum_{j \in N} f_2^j \right).$$

  Via lemma 5 and transitivity, we have gRh.
- ...
- Repeating these steps for all rows (in an upward way), we obtain the desired result.

Second, due to the previous result, we might restrict attention to matrices in the domain $\mathcal{D}_M$ (see axiom SE_M). Define an ordering R° on $\mathbb{R}^m$ such that $uRv \Leftrightarrow u^1 R^\circ v^1$, for all $u, v \in \mathcal{D}_M$; this ordering is well-defined. We show that R° must be the leximin rule; then, R is the Roemer social ordering. The induced ordering R° inherits (from the properties SP, SI^*_M and DC for R) the strong Pareto principle, Suppes indifference (anonymity) and (a slightly stronger version of) Hammond Equity, suitably redefined on the new domain $\mathbb{R}^m$. But then, R° has to be the leximin rule (see, for instance, Bossert and Weymark, 1996, theorem 15). □

B. Characterization of the Van de gaer rule It is easy to verify existence: the Van de gaer rule satisfies all axioms UD, $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$, SP, DC, SI^*_N and SI_M. We prove uniqueness. From lemma 1, R has to satisfy UN. Using UN and transitivity (repeatedly if necessary), we have, for all $a, b \in \mathcal{D}$, that

$$aRb \Leftrightarrow \begin{bmatrix} \frac{1}{n} \sum_{j \in N} a_1^j & \dots & \frac{1}{n} \sum_{j \in N} a_1^j \\ \dots & & \dots \\ \frac{1}{n} \sum_{j \in N} a_m^j & \dots & \frac{1}{n} \sum_{j \in N} a_m^j \end{bmatrix} R \begin{bmatrix} \frac{1}{n} \sum_{j \in N} b_1^j & \dots & \frac{1}{n} \sum_{j \in N} b_1^j \\ \dots & & \dots \\ \frac{1}{n} \sum_{j \in N} b_m^j & \dots & \frac{1}{n} \sum_{j \in N} b_m^j \end{bmatrix}.$$

From here, the proof is the same as in the second part of the proof of proposition 1A. □

C. Independence of the axioms We show that, for both rules, dropping the axioms one by one results in at least one new rule, which satisfies all other axioms.

C1. Roemer rule

- drop $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$: consider the leximin rule applied to vectors

$$\left(u_1^1, u_1^2, \ldots, u_1^n; u_2^1, \ldots, u_2^n; \ldots; u_m^1, \ldots, u_m^n\right) \in \mathbb{R}^{mn}.$$

- drop SP: consider the original "maxmin"-version of the Roemer rule, which maximizes the sum of the mins.
- drop DC: consider the utilitarian rule applied to vectors

$$\left(u_1^1, u_1^2, \ldots, u_1^n; u_2^1, \ldots, u_2^n; \ldots; u_m^1, \ldots, u_m^n\right) \in \mathbb{R}^{mn}.$$

- drop SI^*_M: consider the Van de gaer rule
- drop $\widetilde{\mathrm{SI}}_N$: adapt the Roemer rule using (strictly positive) weights when summing (the mins).

C2. Van de gaer rule

- drop $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$: consider the "extensive" leximin rule, defined in C1.
- drop SP: consider a "maxmin"-version of the Van de gaer rule, which maximizes the minimal sum.
- drop DC: consider the "extensive" utilitarian rule, defined in C1.
- drop SI^*_N: consider the Roemer rule.
- drop SI_M: adapt the Van de gaer rule using weights ($\in \mathbb{N}_0$) when leximinning (the sums).[17]

Proof of proposition 2

First, consider a social ordering R satisfying PD and $\mathrm{I}^{\mathrm{rn}}_{\mathrm{rf}}$. Define a social ordering R° such that for all $u, v \in \mathbb{R}^{m\times n}_{++}$: $\ln u R^\circ \ln v \Leftrightarrow uRv$, with $\ln x$ equal to the matrix $\left[\ln x_i^j\right]$ for $x = u, v$. It is easy to see that R° has to satisfy UD and $\mathrm{I}^{\mathrm{tn}}_{\mathrm{tf}}$. Second, if we impose one of the axioms SP, DC, SI^*_M, SI_M, SI^*_N or $\widetilde{\mathrm{SI}}_N$ on R, then the same axiom must hold for R°. Finally, we can use proposition 1 to characterize R°: depending on whether we additionally impose the set SP, DC, SI^*_M, $\widetilde{\mathrm{SI}}_N$ or the set SP, DC, SI_M, SI^*_N, we obtain that R° must be the Roemer rule (resp. the Van de gaer rule). By definition of both rules and by definition of R°, we get that R must be the Roemer rule (resp. the Van de gaer rule), but applied to logarithmically transformed matrices; this is equal to the product-Roemer rule (resp. product-Van de gaer rule) applied to the original matrices $u, v \in \mathbb{R}^{m\times n}_{++}$. $\square$

[17] As it is rather unusual, weighting the leximin rule with positive integers has to be understood as applying the leximin rule to numbers, which are replicated by its weight. For example comparing $(1, 2, 3)$ with $(2, 1, 3)$ and using $(1, 2, 1)$ as weighting vector, amounts to comparing $(1, 2, 2, 3)$ and $(2, 1, 1, 3)$ via the leximin rule.

Proof of proposition 3

We prove the characterization for the KAS-Roemer family; the other cases are analogous. All orderings in the family satisfy the axioms PD, $\mathrm{I}_{\mathrm{rf}}^{\mathrm{rf}}$, SP, CON, SE_M, SE_N^*, SI_M^*, and SI_N. The other way around, consider a social ordering R satisfying these axioms. Using CON and SE_N^*, this social ordering can be represented by continuous functions f and g, such that for all $u, v \in \mathcal{D}$:

$$uRv \Leftrightarrow f\left(g\left(u^1\right), \ldots, g\left(u^n\right)\right) \geq f\left(g\left(v^1\right), \ldots, g\left(v^n\right)\right).$$

The axioms SP, SE_N^*, SI_M^* and SP, SE_M and SI_N require f and g to be strictly increasing, separable, and symmetric. The axiom $\mathrm{I}_{\mathrm{rf}}^{\mathrm{rf}}$ imposes homogeneity for g and homotheticity for f (because any strictly increasing transformation of f represents the same rule). As a consequence, f is a Kolm-Atkinson-Sen social welfare function defined over vectors $\left(g\left(u^1\right), \ldots, g\left(u^n\right)\right)$, with g strictly increasing, separable, symmetric, and homogeneous, as required. □

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